New electromagnetic conservation laws

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Abstract

The Chevreton superenergy tensor was introduced in 1964 as a counterpart, for electromagnetic fields, of the well-known Bel-Robinson tensor of the gravitational field. We here prove the unnoticed facts that, in the absence of electromagnetic currents, Chevreton's tensor (i) is completely symmetric, and (ii) has a trace-free divergence if Einstein-Maxwell equations hold. It follows that the trace of the Chevreton tensor is a rank-2, symmetric, trace-free, *conserved* tensor, which is different from the energy-momentum tensor, and nonetheless can be constructed for any test Maxwell field, or any Einstein-Maxwell spacetime.

The Bel-Robinson "superenergy" tensor [2, 4] is today a well-known tool in General Relativity. Despite the lack of a conclusive physical meaning, it has been proved as very valuable in many mathematical developments and theoretical applications, see e.g.[13, 15] and references therein. The analogy of many of its properties with those of the energy-momentum tensor of electromagnetic fields is intriguing and certainly suggestive, something which has led many authors to look for similar superenergy tensors of fields other than the graviational one (e.g [7, 13, 15, 16] and references therein). Perhaps the first such attempt appears in the work by Chevreton [7], who introduced a 4-index tensor, similar to the Bel-Robinson one, for the Maxwell field. Chevreton's tensor is quadratic in the derivatives of the electromagnetic 2-form, and divergence-free in the absence of gravitational field, that is, in Special Relativity. This last property does not hold in the presence of curvature [7, 15], possibly leading to the exchange of superenergy between the fields, see e.g. [15, 12].

The purpose of this letter is to prove two apparently unnoticed properties of Chevreton's tensor. In the absence of electromagnetic sources we prove first that the Chevreton tensor is completely symmetric, and then that its trace is divergence-free (or, equivalently, its divergence is trace-free) if either Einstein's vacuum or Einstein-Maxwell's field equations hold.

We will follow the spinor and abstract index notations as defined in [13] and use the signature $(+---)$ (observe that this is opposite to [14, 15]). The metric tensor will be denoted by g_{ab}. The definition of the Chevreton superenergy tensor can today, as is also the case for the Maxwell, Bel [3] and Bel-Robinson tensors, be seen as a construction that comes from the general definition of superenergy tensors in Lorentzian manifolds of arbitrary dimension. This general definition provides an even rank tensor $T_{ab\ldots}\{A_{\ldots}\}$ starting from any arbitrary tensor $A_{ab\ldots}$, the former being quadratic on the latter and called the basic superenergy tensor of $A_{ab\ldots}$. The definition was originally presented in [14] and studied and much developed in [15]. Following this paper, a tensor A_{abc} with the antisymmetry $A_{abc} = A_{a[bc]}$ is called a double (1,2)-form and its basic superenergy tensor $T_{abcd}\{A_{[1],[2]}\}$ can be adequately expressed as [15]

$$T_{abcd}\{A_{[1],[2]}\} = -A_{acf}A_{bd}{}^{f} - A_{adf}A_{bc}{}^{f} + g_{ab}A_{ecf}A^{e}{}_{d}{}^{f} + \frac{1}{2}g_{cd}A_{aef}A_{b}{}^{ef} - \frac{1}{4}g_{ab}g_{cd}A_{efg}A^{efg} \quad (1)$$

which is explicitly independent of the dimension of the spacetime and satisfies $T_{abcd}\{A_{[1],[2]}\} = T_{(ab)(cd)}\{A_{[1],[2]}\}$.

Let $F_{ab} = -F_{ba}$ be a 2-form and consider the (1,2)-form $A_{abc} = \nabla_a F_{bc}$. Its basic superenergy tensor $T_{abcd}\{\nabla_{[1]}F_{[2]}\}$ is given then by (1). We will denote it by E_{abcd} in what follows. If F_{ab} is a Maxwell field, Chevreton's superenergy tensor of the electromagnetic field is defined by [7, 17]

$$H_{abcd} = \frac{1}{2}(T_{abcd}\{\nabla_{[1]}F_{[2]}\} + T_{cdab}\{\nabla_{[1]}F_{[2]}\}) \equiv \frac{1}{2}(E_{abcd} + E_{cdab}) \quad (2)$$

so that it has the obvious symmetries

$$H_{abcd} = H_{(ab)(cd)} = H_{cdab}.$$

It also satisfies the following positive-definite property —called the dominant property [6]—:

$$H_{abcd}u^a v^b w^c z^d \geq 0$$

for all future-directed causal vectors u^a, v^a, w^a and z^a. This follows from (2) and the fact that *any* superenergy tensor has the dominant property [5, 15], as well as any linear combination of them with non-negative coefficients [15, 6]. As mentioned before, in flat spacetime E_{abcd} is divergence-free. Thus, the analogy with the Bel-Robinson tensor is quite stimulating. We are going to make this analogy even stronger in the remaining of this letter.

From now on, we assume that the spacetime is 4-dimensional.[1] Then, as is well-known, the Bel-Robinson tensor is completely symmetric and traceless. We are going to obtain the corresponding properties for the Chevreton tensor. Concerning traces, we first of all have [15]

$$E_{ab}{}^{c}{}_{c} = 0, \quad E^{c}{}_{cab} \neq 0, \quad E^{c}{}_{abc} \neq 0 \tag{3}$$

so that in general the traces of H_{abcd} are non-vanishing. Nevertheless, they satisfy some interesting properties. To prove them, and to obtain the result on the index symmetry, we find it easier to resort to the spinor formalism [13]. We are going to prove that H_{abcd} is completely symmetric if F_{ab} satisfies the source-free Maxwell equations.

The general spinor form of superenergy tensors in dimension 4 was obtained in [5]. For the (1,2)-form $A_{abc} = A_{AA'BB'CC'}$ this is

$$T_{abcd}\{A_{[1],[2]}\} = E_{abcd} = \frac{1}{2}(A_{AB'CE'D}{}^{E'}\bar{A}_{BA'EC'}{}^{E}{}_{B'} + A_{BA'CE'D}{}^{E'}\bar{A}_{AB'EC'}{}^{E}{}_{B'})$$

so that writing F_{ab} in spinor form

$$F_{ab} = \varphi_{AB}\bar{\epsilon}_{A'B'} + \bar{\varphi}_{A'B'}\epsilon_{AB}$$

where $\varphi_{AB} = \varphi_{(AB)} = \frac{1}{2}F_{AE'B}{}^{E'}$ we get for $A_{abc} = \nabla_a F_{bc}$ that

$$A_{AB'CE'D}{}^{E'} = 2\nabla_{AB'}\varphi_{CD}$$

hence its basic superenergy tensor becomes

$$E_{abcd} = 2(\nabla_{AB'}\varphi_{CD}\nabla_{BA'}\bar{\varphi}_{C'D'} + \nabla_{BA'}\varphi_{CD}\nabla_{AB'}\bar{\varphi}_{C'D'}).$$

The general spinor form of the Chevreton tensor is therefore

$$\begin{aligned} H_{abcd} \;&= \nabla_{AB'}\varphi_{CD}\nabla_{BA'}\bar{\varphi}_{C'D'} + \nabla_{BA'}\varphi_{CD}\nabla_{AB'}\bar{\varphi}_{C'D'} \\ &+\nabla_{CD'}\varphi_{AB}\nabla_{DC'}\bar{\varphi}_{A'B'} + \nabla_{DC'}\varphi_{AB}\nabla_{CD'}\bar{\varphi}_{A'B'}\,. \end{aligned} \tag{4}$$

In the absence of electromagnetic sources, the Maxwell equations

$$\nabla^a F_{ab} = 0, \qquad \nabla_{[a}F_{bc]} = 0$$

are equivalent to [13]

$$\nabla^{AA'}\varphi_{AB} = 0 \tag{5}$$

which implies that

$$\nabla_{B'A}\varphi_{CD} = \nabla_{B'(A}\varphi_{CD)} \tag{6}$$

so the Chevreton tensor becomes in this case

$$\begin{aligned} H_{abcd} = &\nabla_{B'(A}\varphi_{CD)}\nabla_{B(A'}\bar{\varphi}_{C'D')} + \nabla_{A'(B}\varphi_{CD)}\nabla_{A(B'}\bar{\varphi}_{C'D')} \\ &+\nabla_{D'(C}\varphi_{AB)}\nabla_{D(C'}\bar{\varphi}_{A'B')} + \nabla_{C'(D}\varphi_{AB)}\nabla_{C(D'}\bar{\varphi}_{A'B')} = H_{(abcd)} \end{aligned}$$

Hence we have proved the following new result:

[1] In this case, an analysis of the Chevreton tensor of null electromagnetic fields can be found in [17].

Theorem 1 *In the absence of electromagnetic sources, the Chevreton tensor is completely symmetric in four dimensions:* $H_{abcd} = H_{(abcd)}$.

Observe that this theorem is valid in arbitrary spacetimes, independently of Einstein's field equations: only the Maxwell equations have been used. We want to remark that Theorem 1 can also be derived as an application of the anti-symmetrization methods developed by Edgar and Höglund [8], which in fact inspired us. Their methods would lead to an alternative tensorial proof of the theorem.

As a corollary of the above theorem, we obtain that the two non-zero traces of E_{abcd} are related by means of

$$\frac{1}{2}E^c{}_{cab} = E^c{}_{abc} = E^c{}_{(ab)c} = H^c{}_{cab} \equiv H_{ab} \tag{7}$$

as follows directly from (2) and the complete symmetry of H_{abcd}. Moreover, due to the first expression in (3) we have also proved that the complete trace of H_{abcd} (or E_{abcd}) vanishes identically in the absence of sources, in other words that the tensor H_{ab} is trace-free:

$$H^a{}_a = 0.$$

We now go on to express the divergence of the Chevreton tensor and show that it is trace-free in the absence of sources. Equivalently, this can be expressed as saying that the trace H_{ab} of the Chevreton tensor, as defined in (7), is divergence-free. Note however that the trace of a superenergy tensor does not need to have the dominant property.

In general there are two divergences of E_{abcd} which are $\nabla^a E_{abcd}$ and $\nabla^c E_{abcd}$. Explicit expressions for them, rather lengthy in general, are given in [15] in terms of the curvature tensor. For the completely symmetric Chevreton tensor H_{abcd} we get of course only one independent divergence. Here we use the spinor form (4) directly to get

$$\begin{aligned}
\nabla^a H_{abcd} \;=\;& \nabla^{AA'}\nabla_{AB'}\varphi_{CD}\nabla_{A'B}\bar{\varphi}_{C'D'} + \nabla_{AB'}\varphi_{CD}\nabla^{AA'}\nabla_{A'B}\bar{\varphi}_{C'D'} \\
&+\nabla^{AA'}\nabla_{BA'}\varphi_{CD}\nabla_{B'A}\bar{\varphi}_{C'D'} + \nabla_{BA'}\varphi_{CD}\nabla^{AA'}\nabla_{B'A}\bar{\varphi}_{C'D'} \\
&+\nabla^{AA'}\nabla_{CD'}\varphi_{AB}\nabla_{C'D}\bar{\varphi}_{A'B'} + \nabla_{CD'}\varphi_{AB}\nabla^{AA'}\nabla_{C'D}\bar{\varphi}_{A'B'} \\
&+\nabla^{AA'}\nabla_{DC'}\varphi_{AB}\nabla_{D'C}\bar{\varphi}_{A'B'} + \nabla_{DC'}\varphi_{AB}\nabla^{AA'}\nabla_{D'C}\bar{\varphi}_{A'B'}
\end{aligned} \tag{8}$$

To study this expression we use commutators and note that there are essentially two types of terms, represented by $\nabla^{AA'}\nabla_{AB'}\varphi_{CD}$ and $\nabla^{AA'}\nabla_{BA'}\varphi_{CD}$. Applying (6) when needed we get, with the notation of [13] and use of (5),

$$\begin{aligned}
&\nabla^{AA'}\nabla_{AB'}\varphi_{CD} = \nabla^{AA'}\nabla_{CB'}\varphi_{AD} \\
&= \nabla_{CB'}\nabla^{AA'}\varphi_{AD} + (\bar{\varepsilon}^{A'}{}_{B'}\Box^A{}_C + \varepsilon^A{}_C\Box^{A'}{}_{B'})\varphi_{AD} \\
&= \bar{\varepsilon}_{B'}{}^{A'}(X^E{}_{CE}{}^F\varphi_{FD} + X^E{}_{CD}{}^F\varphi_{EF}) + \Phi^{A'}{}_{B'C}{}^F\varphi_{FD} + \Phi^{A'}{}_{B'D}{}^F\varphi_{CF}
\end{aligned} \tag{9}$$

and, also using the contraction of (9),

$$\begin{aligned}
&\nabla_A{}^{A'}\nabla_{BA'}\varphi_{CD} = (\nabla_{(A}{}^{A'}\nabla_{B)A'} + \tfrac{1}{2}\varepsilon_{AB}\nabla_E{}^{A'}\nabla^E{}_{A'})\varphi_{CD} \\
&= -(\Box_{AB} + \tfrac{1}{2}\varepsilon_{AB}\nabla^e\nabla_e)\varphi_{CD} \\
&= X_{ABC}{}^F\varphi_{FD} + X_{ABD}{}^F\varphi_{CF} + \varepsilon_{BA}(X^E{}_{CE}{}^F\varphi_{FD} + X^E{}_{CD}{}^F\varphi_{EF}).
\end{aligned} \tag{10}$$

Here $\Phi_{ABA'B'}$ is the trace-free Ricci spinor and $X_{ABCD} = \Psi_{ABCD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})$ with Ψ_{ABCD} being the Weyl spinor and $24\Lambda = R$ the scalar curvature. Substituting (9) and (10) into

(8) and imposing part of the Einstein-Maxwell equations (namely, $\Phi_{ab} = 2\varphi_{AB}\bar{\varphi}_{A'B'}$ but keeping Λ arbitrary) we find after some simplifications and use of (6) and Theorem 1 the following simple expression

$$\begin{aligned}\nabla^a H_{abcd} &= 2\Psi^{EF}{}_{(BC}\varphi_{D)F}\nabla_{E(B'}\bar{\varphi}_{C'D')} + 4\varphi^{FE}\Psi_{FE(BC}\nabla_{D)(B'}\bar{\varphi}_{C'D')} - 18\Lambda\varphi_{(BC}\nabla_{D)(B'}\bar{\varphi}_{C'D')} \\ &+2\bar{\Psi}^{E'F'}{}_{(B'C'}\bar{\varphi}_{D')F'}\nabla_{E'(B}\varphi_{CD)} + 4\bar{\varphi}^{F'E'}\bar{\Psi}_{F'E'(B'C'}\nabla_{D')(B}\varphi_{CD)} - 18\Lambda\bar{\varphi}_{(B'C'}\nabla_{D')(B}\varphi_{CD)}. \end{aligned} \quad (11)$$

Obviously, this expression holds in Einstein spaces too ($\Phi_{ab} = 0$ with any Λ). Since (11) is completely symmetric with respect to spinor indices it is trace-free, that is to say,

$$\nabla^a H_{ab} = 0.$$

Thus, we have the new result

Theorem 2 *The trace of the Chevreton tensor is divergence-free in the absence of electromagentic sources in four dimensions.*

We stress the fact that this result holds (i) for arbitrary Einstein-Maxwell spacetimes with a possible cosmological constant Λ as well as (ii) for any Maxwell test fields in Einstein spaces, including proper vacuum. Therefore, we have constructed a symmetric, trace-free and divergence-free tensor H_{ab} associated to any source-free electromagnetic field. This tensor is quadratic in the derivatives of F_{ab} and therefore it is not related to the energy-momentum tensor (nor to the so-called zilch tensor [11, 10] which is conserved in Special Relativity, see also [1, 9]). Note that by (4) and (7) we have the simple spinor form of H_{ab}

$$H_{ab} = -2\nabla_c\varphi_{AB}\nabla^c\bar{\varphi}_{A'B'}$$

which can be used to derive the above theorem directly. Several convenient tensor forms are

$$\begin{aligned}H_{ab} &= \nabla_c F_{ad}\nabla^c F_b{}^d - \frac{1}{4}g_{ab}\nabla_c F_{de}\nabla^c F^{de} \\ &= \frac{1}{2}\left(\nabla_c F_{ad}\nabla^c F_b{}^d + \nabla_c \overset{*}{F}_{ad}\nabla^c \overset{*}{F}_b{}^d\right) \\ &= 2\nabla_c {}^+F_{ad}\nabla^c {}^-F_b{}^d\end{aligned}$$

where $\overset{*}{F}_{ab}$ is the Hodge dual of F_{ab} and ${}^{\mp}F_{ab} \equiv \frac{1}{2}(F_{ab} \pm i\overset{*}{F}_{ab})$. We wish to remark that the proof of the divergence-free property of H_{ab} is far from obvious when using these tensor expressions.

One can construct conserved currents associated to H_{ab} if Killing or conformal Killing vectors are present. Explicitly, they are given by

$$j^a(\xi) \equiv H^a{}_b\xi^b \implies \nabla_a j^a = 0$$

where ξ^b is any (conformal) Killing vector. These currents, as well as H_{ab}, are invariant against duality rotations[2] ${}^{\pm}F_{ab} \to e^{\pm i\theta}\,{}^{\pm}F_{ab}$ (with constant θ) as follows from the fact that the basic superenergy tensor of any tensor coincides with that of any of its duals [15], or directly from the explicit expressions of H_{ab}. Observe on the other hand that, generically, the dominant property (the dominant energy condition), i.e. that $H_{ab}u^a v^b \geq 0$ for all future-directed causal vectors u^a and v^a, need not be satisfied.

[2] The fitting of this result in Special Relativity with those of [1, 9] is unclear.

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References

[1] Anco S and Pohjanpelto J 2001 *Classification of local conservation laws of Maxwell's equations* Acta Applic. Math. **69/3** 285-327

[2] Bel L 1958 *Sur la radiation gravitationelle* C.R.Acad.Sci.Paris **247** 1094–1096

[3] Bel L 1958 *Introduction d'un tenseur de quatrième ordre* C.R.Acad.Sci.Paris **247** 1297–1300

[4] Bel L 1962 *Les états de radiation et le problème de l'énergie en relativité générale* Cahiers Phys. **16** 59–80 ; (English translation: 2000 Gen. Rel. Grav. **32** 2047–2078)

[5] Bergqvist G 1999 *Positivity of general superenergy tensors* Commun. Math. Phys. **207** 467–479

[6] Bergqvist G and Senovilla J M M 2001 *Null cone preserving maps, causal tensors and algebraic Rainich theory* Class. Quantum Grav. **18** 5299–5325 gr-qc/0104090

[7] Chevreton M 1964 *Sur le tenseur de superénergie du champ électromagnétique* Nuovo Cimento **34** 901–913

[8] Edgar S B and Höglund A 2002 *Dimensionally dependent tensor identities by double antisymmetrization* J. Math. Phys. **43** 659–677 gr-qc/0105066

[9] Fuschich W I and Nikitin A G 1992 *The complete sets of conservation laws for the electromagnetic field* J. Phys. A: Math. Gen. **25** L231-L233

[10] Kibble T W B 1965 *Conservation laws for free fields* J. Math. Phys. **6** 1022-1026

[11] Lipkin D M 1964 *Existence of a new conservation law in electromagnetic theory* J. Math. Phys. **5** 696-700

[12] Lazkoz R, Senovilla J M M, and Vera R 2003 *Conserved superenergy currents* preprint gr-qc/0302101

[13] Penrose R and Rindler W 1986 *Spinors and spacetime* vols 1-2 Cambridge Univ. Press

[14] Senovilla J M M 1999 *Remarks on superenergy tensors* in "Gravitation and Relativity in General. Proc. of the Spanish Relativity Meeting in Honour of the 65th Birthday of L Bel", ed. J Martín *et al.* (Singapore, World Scientific), pp. 175-182; gr-qc/9901019

[15] Senovilla J M M 2000 *Super-energy tensors* Class. Quantum Grav. **17** 2799–2841 gr-qc/9906087

[16] Teyssandier P 2001 *Can one generalize the concept of energy-momentum tensor?* Ann. Foundation L. de Broglie **26** 459-469; (see also gr-qc/9905080)

[17] Wallace G and Zund J D 1988 *Electromagnetic theory in General Relativity VII: the superenergy tensor* Tensor N S **47** 179-188